Managing Sponsor Risk in Pension Plans: Dynamic Strategies vs. Pension Assurance

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WORKING PAPER.

First version: March 2011. This version: April 30, 2012

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I thank Dirk Broeders, Jakša Cvitanić, Stéphane Gregoir, Con Keating, Raman Uppal, Vincent Milhau, and especially, Pierre Mella-Barral, for very useful comments.
A previous version of this paper called 'Managing Sponsor Risk in Pension Plans' was disseminated and presented at EFFAS summer school in July 2011.

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Abstract

Defined-benefit (DB) pension funds, often underfunded, rely on the legal obligation of their sponsor to secure pension rights. This paper firsts in identifying the optimal pension funds portfolio, while considering the risk on the sponsor's guarantee and practical limitations due to the liquidity of the sponsor's shares and to the fund's leverage constraints. It also firsts in disentangling risk-management from risk-shifting incentives in pension plans. Non-contractible plans are Third-Best, but Pension Indemnity Assurance, a non-redundant asset and contract, reduces the risk-shifting incentives from each party to the pension contract and fully mitigates sponsor risk.

Keywords: Pension Fund, Asset Allocation, Asset and Liability Management, Defined-Benefit, Risk-Management, Risk-Shifting, Sponsor Put, Sponsor Risk, Portfolio Insurance.

JEL: G11, G23, G31, G32, C61.

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1 Introduction

1.1 Organisation of Defined-Benefit Plans

A pension fund pools the assets bought with the contributions to a pension plan for the exclusive purpose of financing pension plan benefits. It is legally independent from the sponsor but financially dependent because of the guarantee provided by the sponsor. Plan members have a legal right to all assets in plans that are legally incorporated, and they have rights to assets that back the termination value (also called the *buy-out value*) of the pension rights in pension trusts. In addition, members benefit from the sponsor's guarantee to cover any pension plan deficit: unlike defined contribution (DC) plans, DB plans are designed to guarantee a minimum retirement income.

Management of the pension plan – both of plan assets and of contributions – is the responsibility of a board made up of the sponsor, employee representatives, the financial institution that manages the plan and an independent advisor (the pension actuary). The overriding objective of the board is to provide a secure source of retirement income and it must usually act in the sole interest of plan members. This may of course conflict with the trustees' duty of loyalty (or good faith) to the sponsor which implies that trustees cannot capriciously raise the sponsor's pension costs.

In the UK and in the US, pension funds are explicitly allowed to be permanently underfunded until pensions are paid, and the sponsor's pension debt, even if partly recognised in its accounts, is unfunded too; the sponsor is not required to set aside the present value of the benefit.[1] DB is by definition always synonymous with an unfunded sponsor guarantee, because of the value of possible recovery contributions.[2] Full funding requirements imply no recovery contributions from sponsors and a shift from DBs to DCs.[3]

So in initially overfunded pension plans relying solely on funding requirements and risk management to secure pension rights means losing the value of the guarantee, while underfunding of pension plans combined with risky unfunded guarantee means highly risky pension rights.

The first contribution of this paper is to show that in complete markets and without risk-shifting, a unique optimal terminal payoff exists, and to study the implementation trading sponsor shares.

The second contribution is the analysis of risk shifting incentives from either sponsor or pension fund. Risk-shifting is analytically disentangled from risk-management. Risk-shifting incentives from both parties is attenuated by a non-redundant contract, pension indemnity insurance (PIA), which also fully mitigates sponsor risk, increasing the pension fund utility by an amount equal to the value of the guarantee; under full information, an improved fund behaviour results in gains equivalent to the total costs of running a plan; with asymmetry of information, gains can be in order of 20% over 10 years.

[1] In Germany, the pension debt can be entirely backed by machinery and equipment.

[2] Thus, a DB pension fund always is underfunded relative to the risk-neutral value of pension benefits, and this, even when there are more funds than necessary to pay for the minimum guaranteed pension liability.

[3] Financial economists have claimed that the present value of the future benefits should be set aside as cash – in the pension fund to avoid risk-shifting from the sponsor – which secure the value of the guarantee and pension rights but also makes the plan a DC.

1.2 Literature

The risk of sponsors leaving an underfunded pension fund, or *pension put* as coined by Sharpe (1976), has been studied from the point of view of the public pension insurance company that takes over (usually the US PBGC). Treynor (1977), Marcus (1985), Pennacchi and Lewis (1994), and Broeders and Chen (2011) price the value that can be lost with a defaulting sponsor and allow for a realistic valuation of the pension rights. But in all these papers, the pension fund's investment strategy is taken as externally given, rather than optimally defined with a utility function.

From the point of view of the fund, the risk that the sponsor walks away leaving diminished pension rights is referred to as *sponsor risk*. The literature usually assumes that trustees – whether independent or under sponsor's control – solely control the volatility of investments, which does not allow hedging sponsor risk nor disentangling risk-shifting from risk-management incentives. Then the optimisation program cannot always be solved and empirical tests evaluate whether data shows evidence of *either* risk-shifting *or* risk-management, whereas *both* can take place simultaneously.

Jensen and Meckling (1976) study risk-shifting and contracting, but their focus is essentially on outside debt and equity instruments. Risk-shifting incentives from the sponsor have been studied (*e.g.*, Sharpe 1976, Rauh 2009), but not the way to remedy these. Risk-shifting from the pension fund has similarities with risk-shifting from a fund manager (Carpenter, 2000), but as the pension fund receives the entire value of the assets in the plan when it ends over-funded the type of contract needed with pension funds is entirely different from that needed with a fund manager.

1.3 Organisation of the Paper

This paper studies optimal investment strategies and pension indemnity insurance (PIA) within existing DB pension contracts: the institutional framework is considered as given. Sponsor make their funding decision in equilibrium requiring that the total cost of pension is equal to the agreed risk-neutral value of pension payments. We disentangle the conflicting incentives between loyalty to the sponsor and maximisation of the participants' utility and show that without management of sponsor risk, the trustee's maximisation program is generally not feasible without losing the value of the guarantee.

For pension funds, a non-contractible sponsor requires external instruments and not solely dynamic strategies; risk-shifting from the pension fund results in low initial funding; we study a PIA contract that trivially solves the issue of non-contractibility of the sponsor and is designed to optimally diminish the risk-shifting incentives and associated costs in pension funds.

Section 2 defines notations and the problem. Section 3 studies the 'First-Best' strategy with contractible parties. Section 4 studies implementation and limitations using sponsor shares. Section 5 shows that PIA solves traditional DB plans optimisation problem even with risk-shifting. Section 6 shows that non-contractible (hybrid) plan are 'Third-Best'. Section 7 shows that PIA results in a 'Second-Best' outcome.

2 The Economy and the Set-Up of the Model

2.1 Definitions and the Economy

The assumptions chosen are typical of models used in the valuation of pension liabilities (Broeders and Chen, 2011) and they extend the notations of portfolio insurance (Basak and Shapiro, 2001) by considering that the sponsor's wealth is stochastic, so that it cannot fulfill its commitment to cover future pension deficits with certainty out of its balance sheet.

We consider a continuous-time, finite-horizon $[0,T]$ economy prices are exogenous to the pension fund. Uncertainty is represented by a filtered probability space $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}, \mathbb{P})$, on which an n-dimensional Brownian motion W is defined. All stochastic processes are assumed to be adapted to $(\mathfrak{F}_t, t \in [0,T])$, the augmented filtration generated by W. The market is complete and all stated (in)equalities involving random variables hold $\mathbb{P}$-*almost surely*.

The vector of traded risky assets P follows the process: $dP_t = diag(P_t)\cdot(\mu_t \cdot dt + \sigma_t \cdot dW)$ where the vector μ_t represents the risk premium and the matrix σ_t stacks the volatility of the assets, subject to suitable regularity conditions. The pension fund guaranteed liability L_t has volatility σ_{Lt}. We suppose that there is an efficient market to continuously transfer all risks including longevity.[4] We denote θ the risk premium process, $\theta_t = \sigma_t^{-1}\cdot(\mu_t - r_t)$.

Dynamic market completeness (under no-arbitrage) implies a unique state

[4]An alternative representation also useful is that plan members receive an annuity (without longevity risk) starting at time T, and that it is required that the pension fund be fully funded at $t = T$ – at which moment any recovery contribution must happen.

price density process, or stochastic discount factor, Z with diffusion $dZ_t = -Z_t \cdot (\theta_t' \cdot dW + r_t \cdot dt)$ and value $Z_t = \exp(\int_0^t -\theta_t' dW - \int_0^t [r_t + \frac{\|\theta_t\|^2}{2} dt])$

In complete markets, there is also a unique martingale measure $\mathbb{Q}$ defined by $\zeta_t = e^{\int_0^t (r_t \cdot dt)} \cdot Z_t$ with respect to the historical probability measure $\mathbb{P}$. $\mathbb{Q}$ has Radon-Nikodym density $\zeta_t = \exp(-\int_0^t \theta_t' dW - \frac{1}{2}\|\theta_t\|^2 dt)$. As the utility is on the funding ratio, *i.e.*, on a measure of wealth where the liability serves as a numeraire, we use, by default, the corresponding liability forward martingale measure $\mathbb{Q}_L$ where the liability price L_t plays the role of the T-maturity bond price in the T-forward probability measure used in bonds derivatives pricing.[5]

$F_t = \frac{A_t}{L_t}$, the funding ratio at time $0 \leqslant t \leqslant T$, is a martingale under $\mathbb{Q}_L$, so that in all equations, the budget constraint on A_T is written instead as a budget constraint on F_T under $\mathbb{Q}_L$, without discounting. The sponsor has a net asset value Y denoted by $G = \frac{Y}{L}$ in the numeraire of the liability. G has volatility $\sigma_G = \sigma_Y - \sigma_L$. The share value S will be determined endogenously. Trustees must secure an important fraction k of L_T, usually $k = 1$.

It is convenient to distinguish cp_T the conditional time-T recovery cash contributions from the sponsor and ins_T the proceeds from the protection from financial markets against sponsor risk. Both are fractions of L.

[5] Geman et al. (1995) showed that one can change numeraire to any strictly positive 'asset' L and associate to it an equivalent probability measure. The numeraire invariance principle states that the pricing of securities or payoffs is not affected by changes of numeraire: when L_t is the new numeraire, any asset A (divided by L) is a martingale under $\mathbb{Q}_L$, *i.e.*, $E_t^{\mathbb{Q}_L}[\frac{A_T}{L_T}] = \frac{A_t}{L_t}$. The Girsanov theorem describes the needed transformations in the new probability measure: applying Ito's Lemma on $\frac{A_t}{L_t}$, one can check that the new drift of asset A must be $\mu_A^{\mathbb{Q}_L} = \mu_A^{\mathbb{Q}} + < \frac{dA}{A} \cdot \frac{dL}{L} >$, with $< \frac{dA}{A} \cdot \frac{dL}{L} > = \sigma_A \cdot \sigma_L' = \rho_{AL} \cdot ||\sigma_A|| \cdot ||\sigma_L||$ and $\mu_A^{\mathbb{Q}} = r$. A reference funding ratio can be too used as a numeraire, see Appendix A.

While the funding ratio, asset value and portfolio weights can be expressed as a function of the *s.d.f* Z and the liability value L, we introduce specific notations for relevant reference funding ratios – thus using an asset-space representation of the problem, not solely its classical *s.d.f* representation.

Prefunded strategy: F^{pref} denotes the optimal funding ratio of a pension fund with initial funding ratio $H_0 = F_0 + g$ and no further support from the sponsor. $U\left(F_T^{pref}\right)$ is the supremum of the attainable utility, as it benefits unconditionally and in advance from the agreed value of the guarantee g.

$\tilde{F} = F + ins$ includes the value of the insurance against sponsor risk. F^u denotes the optimal funding ratio of a CRRA investor with initial funding k, relative risk aversion γ but without the minimum funding constraint nor sponsor support. m, the participation to F^u, is a positive number chosen so that the budget constraint is satisfied. H_T represents the 'total' terminal funding ratio after receiving at date T recovery cp_T and ins_T.

π denotes the portfolio weights.[6] $[a \vee b] = max(a,b)$ and $[a \wedge b] = min(a,b)$.

2.2 The pension Fund Optimisation Problem

Problem 2.1. *The pension fund receives, in full, the terminal value of the funding ratio plus recovery contributions from the sponsor. The pension fund seeks to maximise the utility*[7] *from the* total *terminal funding ratio* H_T*:*

[6] Superscripts $*, u, pref$ and $\tilde{}$ are used on A and π to denote assets and portfolio weights corresponding to superscripted F.

[7] The pension fund has a CRRA-type utility frequently used to describe preferences on wealth – note that pension funds store retirement wealth of plan members. Its preferences are expressed on the terminal funding ratio, as liabilities are an explicit reference for

$$U(H_T) = \begin{cases} -\infty & if \quad H_T < k \\ \frac{{H_T}^{1-\gamma}}{1-\gamma} & if \quad H_T \geqslant k \end{cases} \tag{1}$$

This stylisation is representative of continental European hybrid plans.

Lemma 2.2. *UK-US traditional DBs as a simpler special case.*
In a traditional DB members only receive the guaranteed liability or less if a bankrupt sponsor leaves an underfunded plan. Maximising (1) *with any value of* γ *also maximises the utility of participants, because it guarantees with certainty the payment of the pension liability (and maximises it). So trustees in traditional DBs have little if no risk-shifting incentives.*

2.3 The Sponsor's Funding Problem

The value (total risk-neutral price) of the retirement contract is linked to the employment contract and defined in advance. When the fund is subject to a fair value constraint that we equivalently call a no-risk-shifting condition, cp_T and ins_T are subject to: $\mathbb{E}_0^{\mathbb{Q}_L}[cp_T + ins_T] \leqslant g$ (2)
where g is the maximum agreed value of the guarantee as a percentage of the liability.[8] When an incentive contract is given to non-contractible funds, its cost is part of the fair value of the pension contract.

pension trustees. Since the overarching purpose of fund trustees is to protect pension rights, one needs to embed an aversion to underfunding in the utility or plan participants would not choose a DB plan with a guarantee but rather a DC plan. The risk aversion is infinite here. Solutions for alternative utility functions are available upon request.

[8]In general, a fair contract requires equality yet the fund must still 'saturate' the value of the guarantee – traditional portfolio insurance secures the minimum funding ratio while sacrificing the value of the guarantee – hence the inequality in (2).

3 ALM for First-Best Contractible Plans

3.1 Contractible Generalised Portfolio Insurance

This section assumes an ideal situation where both the sponsor's action and that of the pension fund can be contracted.

Contractible Generalised Portfolio Insurance

Theorem 3.1. *Generalised portfolio insurance theorem.*

In the region $F_T^* > k$, the optimal terminal payoffs F_T^* and H_T^* are equal to the optimal F_T^{pref} of the prefunded pension fund (with initial budget constraint $F_0 + g$). In the region where $F_T^* \leq k$, the pension fund is indifferent to the payoffs as long as the risk to conditional sponsor contributions are hedged.

$$H_T^* = F_T^{pref} = \begin{cases} m^* \cdot F_T^u & \text{if} \quad m^* \cdot F_T^u > k \\ k & \text{if} \quad m^* \cdot F_T^u \leqslant k \end{cases} \tag{3}$$

$$(F_T^*, ins_T^*) = \begin{cases} (F_T^* = F_T^{pref} = m^* F_T^u \text{ and } ins_T^* = 0) & \text{if} \quad m^* \cdot F_T^u > k \\ \text{jointly defined } (F_T^* \leqslant k, ins_T^*) & \text{if} \quad m^* \cdot F_T^u \leqslant k \\ \qquad \text{such as} & ins_T^* = k - F_T^* - m_G \cdot G_T \\ \qquad \text{and (if sponsor pays } ins\text{)} & \mathbb{E}_0^{\mathbb{Q}_L}[k - F_T^*]^+ = g \end{cases}$$

Remark 3.2. Insurance and the ex-ante participation constraint.

To fulfill the ex-ante employee participation constraint, the sponsor needs to provide or pay for an insurance on future recovery contributions. In complete markets, the insurance component can be replicated as a bivariate put option with the sponsor's net asset value as a driver and a traditional portfolio insurance strategy as another.[9]

Proof. Proving theorem 3.1 simply requires applying the verification theorem for stochastic processes and verify that the terminal total payoff of the class of strategies is equal to the optimal F_T^{pref} (see A.2 on page 37). In particular, underfunding is needed with a positive probability so that recovery contributions have a positive value; naturally, protection against the risk that the sponsor cannot make good on plan shortfalls is needed. □

3.2 Numerical Analysis

The illustration assumes constant interest rates and constant volatilities (so assets and liabilities are log-normal).[10]

Portfolio Weights: a Four-Fund Separation Theorem

Theorem 3.3. *(Four-fund separation theorem) With a deterministic parameter set, for any optimal strategy the asset allocation involves four funds: the*

[9]Traditional DB plans could have a univariate protection against sponsor risk as their unique asset.

[10]In practice, the replication of the liability requires a thorough definition of the liability process and, in general, stochastic interest rates. Calibration is not the focus here, so the liability process is left undefined and notations are kept simple. Globally speaking, in complete markets a constant parameter set is not a loss of generality but only simplifies notations.

performance seeking portfolio, the liability hedging portfolio, the cash-account and a protection against sponsor risk that can be seen as a dynamic short position on the sponsor's shares when the market is complete.

This theorem has fundamental implications for risk management of corporate pension plans. Its proof follows naturally from the optimal payoff.

Proof.

The optimal strategy is not uniquely defined in the region where $m^* \cdot F_T^u < k$, yet for any optimal strategy, the existence of at least four funds can be proved by looking at portfolio weights at the boundary of the values of the main drivers of the investment strategy.

- As any strategy F^* embeds F^u, it comprises three funds, π_L the liability-hedging portfolio, π_{PSP} the tangency performance-seeking portfolio and the cash-account in which the the remainder – unallocated wealth – is invested in the cash-account. Formally, for $t \to T$ and $F_t^* \gg k$, $F_t^* \to m^* \cdot F_t^u$, then $\pi^* \to \pi^u = \left(1 - \frac{1}{\gamma}\right)(\sigma')^{-1}\sigma_L + \frac{1}{\gamma}(\sigma\sigma')^{-1}(\mu - r1) = \left(1 - \frac{1}{\gamma}\right)\pi_L + \frac{1}{\gamma}\pi_{PSP}$,

- And, of course, ins_T and ins_t have negative exposure to G: if ins has no exposure to G then the value of recovery contributions is nil and the pension fund is of the DC type; if there are recovery contributions, then for $F_T^* < k$, for $G_T \to \infty$, $ins_T \to 0$ and for $G_T \to 0$, $ins_T \to [k - F_T^*]^+$.

Replicating ins requires a short position on the the portfolio that replicates exposure to G, formally speaking $(\sigma')^{-1}\sigma_G$, which can be implemented with the sponsor's shares, see page 20. This implies a fourth fund.[11] □

[11] Of course, if one cannot trade sponsor's (related) instruments, then insurance against

Quantitative Illustration

We numerically compare these with the following example: the pension fund has a bullet liability with 10 years time to maturity; the agreed value of the guarantee is 0.12; the minimum funding ratio below which utility is $-\infty$ is $k = 1$; the pension fund has risk aversion $\gamma = 3.5$; the volatility of the unconstrained strategy F^u is $\sigma_u = 0.1$, and its drift is $\mu_{F^u} = 3.5\%$.

Example 3.4. *A simple implementation.*

Generalised portfolio insurance strategies involve a traditional 'naked' portfolio insurance strategy F^* defined by m^* and $\underline{k}$ plus a bivariate insurance component ins. the following steps:

- *Design a strategy* $F_T^* = \begin{cases} m^* \cdot F_T^u & \text{if} \quad m^* \cdot F_T^u > k \\ \underline{k} & \text{if} \quad m^* \cdot F_T^u \leqslant k \end{cases}$ *where* m^* *is that of the prefunded strategy and* $\underline{k}$ *is such that* $\mathbb{E}[k - \underline{k}]\mathbb{N}(d_-) = g$. [12]

- *Insure sponsor risk with* $[k - F_T^* - m_G \cdot G_T | F_T^* < k]^+$ *for some* $0 < m_G \leqslant 1$.

Remark 3.5. Stochastic floor representation of the strategy.

The combination of the payoff from traditional portfolio insurance (PI) and that from insurance against sponsor risk in example 3.4 leads to:

$\forall F_T^* < k$, $\tilde{F}_T^* = F_T^* + ins_T = [F_T^* \vee (k - m_G \cdot G_T)] = [\underline{k} \vee (k - m_G \cdot G_T)]$.

Thus $\tilde{F}_T^* = [\underline{k} \vee (k - m_G \cdot G_T)] \cdot 1_{m \cdot F_T^u < k} + m \cdot F_T^u \cdot 1_{m \cdot F_T^u > k}$

sponsor risk will only be approximate and pension rights of underfunded pension plans cannot be fully secured – still in general hedging will generate significant utility gains.

[12] This is equivalent to choosing m^* such that $H_0 = F_0^{pref} = \mathbb{E}^{\mathbb{Q}_L}[k \vee (m^* \cdot F_T^u)] = m^* \cdot F_0 \cdot \mathbb{N}(d_+(.)) + k \cdot \mathbb{N}(-d_-(.))$ and $\underline{k}$ such that $F_0 = m \cdot F_0 \cdot \mathbb{N}(d_+(\dots)) + \underline{k} \cdot \mathbb{N}(-d_-(\dots))$.
N.B.: the smooth $[\underline{k} \vee m \cdot F^u]$ is more natural in practice. The digital strategy detailed here compares straightforwardly with the non-contractible strategy in Section 6 on page 23.

Portfolio Weights of $\tilde{F} = F + ins$

- Replicating F_T^* involves exposure to the liability hedging portfolio (L if traded, $\pi_L = (\sigma')^{-1}\sigma_L$ otherwise) and to the performance-seeking portfolio $\pi_{PSP} = (\sigma \cdot \sigma')^{-1} \cdot (\mu - \mathrm{r}1)$. One can check that the weights π^* are:

$$\pi_t^* = \left(1 - \frac{1}{\gamma}\left(1 - \frac{\underline{k} \cdot \mathbb{N}(d_-)}{F_t^*}\right)\right)\pi_L + \frac{1}{\gamma}\left(1 - \frac{\underline{k} \cdot \mathbb{N}(d_-)}{F_t^*}\right)\pi_{PSP}$$

- Replication of the insurance.[13]

$$\begin{aligned} ins_t &= \mathbb{E}_t^{\mathbb{Q}_L}[ins_T] = \mathbb{E}_t^{\mathbb{Q}_L}[(k - F_T^* - m_G \cdot G_T) \cdot \mathbb{1}_{m^* \cdot F_T^u < k}]^+ \\ &= \mathbb{E}_t^{\mathbb{Q}_L}\left[(k - \underline{k} - m_G \cdot G_T)^+ \cdot \mathbb{1}_{m^* \cdot F_T^u < k}\right] \end{aligned} \quad (4)$$

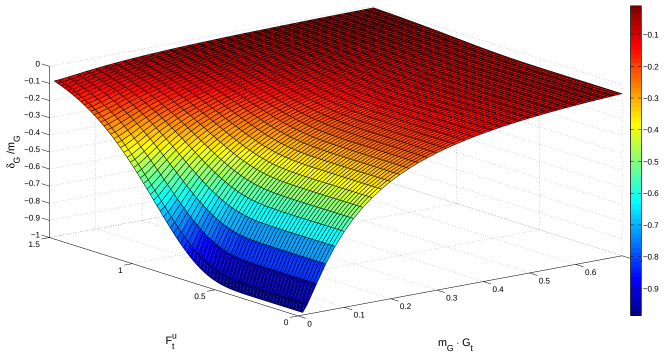
This payoff (4) is a bivariate option and can be expressed in closed form: $ins_t = (k - \underline{k}) \cdot B(-d_-^{F^u}, -d_-^G, \rho) - m_G \cdot G_t \cdot B(-d_+^{F^u}, -d_+^G, \rho)$, with notations: $d_-^X = \frac{\ln(X(t)/K) - \frac{1}{2}\sigma_X^2 \cdot (T-t)}{\sigma_X\sqrt{T-t}}$ where $X = m^* F_t^u$ or $m_G \cdot G_t$ and $K = k$ or $(k - \underline{k})$. Last, $d_+^G = d_-^G + \sigma_G\sqrt{T-t}$ and $d_+^{F^u} = d_-^{F^u} + \rho \cdot \sigma_G\sqrt{T-t}$

$\frac{\partial ins(G, F_t^u, t)}{\partial G_t} = -m_G \cdot B(-d_+^{F_t^u}, -d_+^{G_t}, \rho)$ and replicating ins involves a short position in the replicating portfolio for G_t.[14] ins_t involves additional control on F so $\tilde{F}$ reverts much more quicker to the LHP when F_t^u (thus F_t^*) and G_t are low. Conditional weights of F, $\tilde{F}$ are ins are shown on figures 1 to 6.

[13] In general, ins will be the sum of a spread option and of a bivariate put option. Here, as F_T has a digital component, ins is a simple bivariate put.

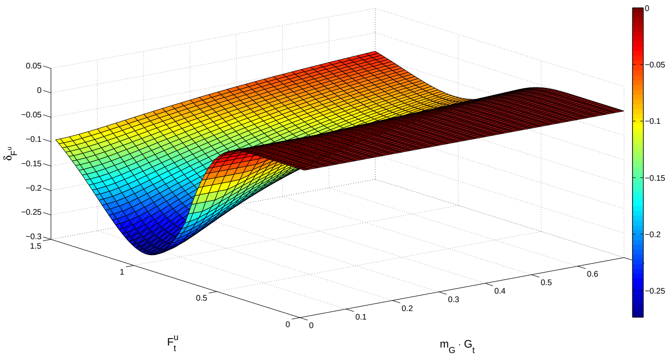
[14] The delta of ins_t *w.r.t.* F_t^u can also be derived in closed form as the conditional probability of each variable in a bivariate law is normal, with $\mu_{x|y=y_0} = \mu_x + \rho\sigma_x\frac{y-\mu_y}{\sigma_y}$ and $\sigma_{x|y=y_0} = \sigma_x \cdot \sqrt{1-\rho^2}$. So, one can perform $\frac{\partial ins_t}{\partial F_t^u} = \frac{\partial ins_t}{\partial d_\pm^{F_t^u}} \cdot \frac{\partial d_\pm^{F_t^u}}{\partial F_t^u}$. Stable numerical derivation is also available from any mathematical software.

Figure 1: Portfolio weights: ins_t involves a short position in $m_G \cdot (\sigma')^{-1}\sigma_G$



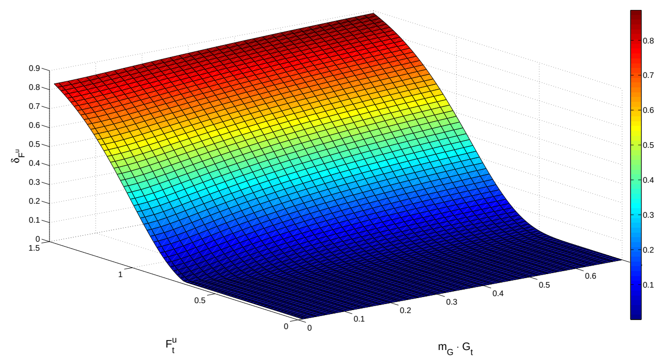
The short position in $(\sigma')^{-1}\sigma_G$ increases to m_G both when G_t or F_t^* fall. For easier reading, the F^u multiplier is set to $m^* = 1$, and G scales are a function of $m_G \cdot G$.

Figure 2: Portfolio weights: ins_t involves additional control on F_t^u



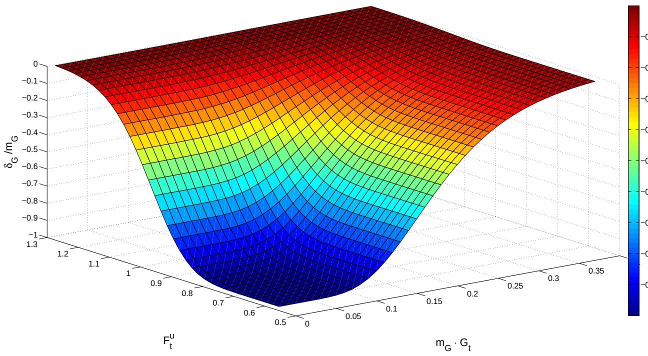
ins_t involves greater control on F_t^u, especially when F_t^* approaches k and for smaller values of $m_G \cdot G_t$.

Figure 3: Portfolio weights: allocation to F_t^u of $\tilde{F}_t = F_t + ins_t$



On the whole, the allocation to risky assets falls at a greater rate when the sponsor's health is weak.

Figure 4: One year portfolio weights: short position in $m_G \cdot (\sigma')^{-1} \sigma_G$



As time to maturity shrinks, the action is more concentrated around $F_t^* = k = 1$ and $m_G \cdot G_t = k - \underline{k} = 0.22$

Figure 5: One year portfolio weights: ins_t involves additional control on F_t^u

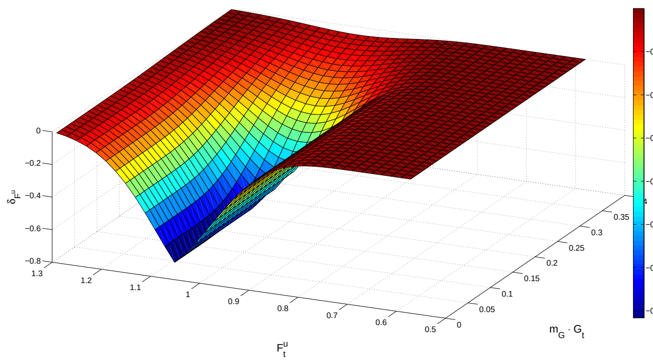
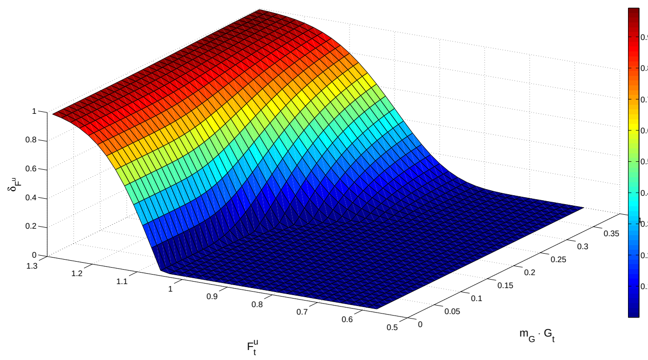


Figure 6: One year portfolio weights: allocation to F_t^u of $\tilde{F}_t = F_t + ins_t$



For $m_G \cdot G_t$ close to 0, the strategy resembles traditional portfolio insurance (but at that point the short position on G has already allowed the funding ratio to rise above $k = 1$ and (G)PI has become feasible).

4 Implementation with Sponsor Shares

4.1 The Value of the Sponsor's Shares

As shown in theorem 3.1, the strategy in the region where $F_T^* < k$ is only characterised by a budget constraint, and ins is not uniquely defined: the limiting condition $G_T \geqslant (k - \tilde{F}_T^*) = k - F_T^* - ins_T$ requires $ins_T \geqslant k - F_T^* - G_T$. $ins_T = [k - F_T^* - m_G \cdot G_T]^+ \cdot 1_{F_T^* < k}$ for some $0 < m_G \leqslant 1$ fulfills this condition.[15]

The exposure to G can[16] be replicated by a dynamic exposure to the sponsor's shares S which reflect the riskiness of the sponsor.[17] For consistency with other notations S is directly denominated in the liability.[18] There is a unique infinitely divisible share which value is the market value of the firm.

We assume that markets close at T_+, so the sponsor's share price at $t = T_+$ is the sponsor's net asset value at T minus the recovery contributions paid out to the pension fund: $S_T = G_T - cp_T$. Thus the firm's market value $S_t = \mathbb{E}_t^{\mathbb{Q}_L}[G_T - cp_T]^+$,[19] reads:[20] $S_t = G_t - \mathbb{E}_t^{\mathbb{Q}_L}[k - \tilde{F}_T^*]^+ = G_t - \mathbb{E}_t^{\mathbb{Q}_L}[k - F_T^*]^+ + ins_t$.

[15] Of course, $m_G > 0$ is necessary for underfunding to be possible; a more general formula is $ins_T = [k - F_T^* - [m_G \cdot G_T - \delta]^+]^+ \cdot 1_{F_T^* < k}$ for some $\delta \geqslant 0$. Its study is left aside for brevity.

[16] As we will see, hedging by the fund modify the hedging properties of S.

[17] In incomplete markets S hedges G at the point of bankruptcy: if the sponsor cannot pay mandatory recovery contributions, it goes bankrupt and its share value falls to zero.

[18] Thus there is liability risk in S. For practical purposes, when relying on the actually traded sponsor shares, not on single futures or liability-denominated shares, one will need to hedge liability risk, which always includes interest rate risk.

[19] As the sponsor has no other debt than pension debt, it makes no difference if the markets close at a later period.

[20] In the current numerical application, $\mathbb{E}_t^{\mathbb{Q}_L}[k - F_T^*]^+ = H_t^* - F_t^* = (k - \underline{k}) \cdot \mathbb{N}(-d_-)$. Because the payoff in the current implementation is binary, the same formula appears in the non-contractible case, but d_- is computed with a higher threshold $F_T^+ > k$, and with $\underline{k} = 0$.

4.2 Hedging with Sponsor Shares

In complete markets, one can define a replicating portfolio $(\sigma')^{-1}\sigma_G$ for the sponsor's net asset value. In practice, it suffices that its shares are traded.[21]

Since $S = G - \mathbb{E}_t^{\mathbb{Q}_L}[k - F_T]^+ + ins$ (and $\mathbb{E}_t^{\mathbb{Q}_L}[k - F_T]^+$ is independent of G), we have $\frac{\partial S}{\partial G} = 1 + \frac{\partial ins}{\partial G}$, thus $\frac{\partial ins}{\partial S} = \frac{\partial ins}{\partial G} / \frac{\partial S}{\partial G} = \frac{\partial ins}{\partial G} / (1 + \frac{\partial ins}{\partial G})$. Since $B(.,.) \leqslant 1$, $\lim_{t \uparrow T, G_t \downarrow 0, F_t^u \downarrow 0} (\frac{\partial ins}{\partial S}) = -\frac{m_G}{1-m_G}$. The terminal payoffs to the shareholders are:

$$S_T = \begin{cases} G_T & \text{if} \quad m^* \cdot F_T^u > k \\ G_T - (k - \underline{k}) & \text{if} \quad m^* \cdot F_T^u \leqslant k \text{ and } m_G \cdot G_T > (k - \underline{k}) \\ (1 - m_G) \cdot G_T & \text{if} \quad m^* \cdot F_T^u \leqslant k \text{ and } m_G \cdot G_T < (k - \underline{k}) \end{cases} \tag{5}$$

$0 < m_G < 1$ is required for implementation.[22] m_G must be chosen sufficiently small to ensure smooth market conditions (one must be able to implement the maximum hedging demand $\frac{m_G}{1-m_G}$, accounting for the liquidity of the sponsor's listed market value [23] and for the fund's leverage constraints).

Remark 4.1. A put option bought by the pension fund (to hedge sponsor risk) has a different price than a put option bought by an unrelated party because hedging changes S – the fund never causes the sponsor's bankruptcy.

[21]One can also implement an optimal insurance program similar to (4) on page 16 but *directly* based on sponsor's shares. – then lesser assumptions are required to price shares.

[22]For $m_G = 1$, when $G \downarrow 0$, $\frac{\partial S}{\partial G} \downarrow 0$ and it becomes impossible for the pension fund to short a sufficient number of shares to short: $\frac{\partial ins}{\partial G} \downarrow -\infty$: as the rise in ins fully offsets the fall in G, the fall in the pension debt offsets the fall in G and S becomes locally insensitive to G.

[23]If m_G is in the neighbourhood of 20%, then it also means that the maximum pension debt $k - \underline{k}$ must be 5 times lesser than the net asset value or market value the sponsor when the pension plan is designed and the initial contribution is made.

5 Sponsors' Risk-shifting in Traditional DBs

Sponsors may perform risk-shifting, as is well known since Jensen and Meckling (1976, p49): 'when the firm has difficulty meeting some of its obligations (...) the issue of the priority of those claims can pose serious problems'. Traditional DBs are well-fit to isolate and analyse sponsor's risk-shifting, because trustees have no risk-shifting incentives: their sole concern is to receive their promised pension payment, see Lemma 2.2 on page 11.

To protect pension rights,[24] traditional DB funds can involve as unique asset a univariate insurance (or put option) against the risk that the sponsor's share value or net asset value is lower than the terminal value of the pension liability, *i.e.*, $ins_T = [k - m_G \cdot G_T]^+$.[25]

This option can be efficiently replicated by a dynamic trading strategy on the sponsor's shares in an ideal world with complete markets and contractible agents, not when sponsor's risk-shifting means that the net asset value G can suddenly become riskier (greater leverage) or disappear (cash-sweeping facility or re-prioritisation of rights). With the disappearance of smooth conditions on S and G, external instruments are needed to protect the fund.

Insurance contracts must be preferred to complex OTC put options both for implementation and incentives. PIA facilitates implementation because risk absorption is not present with PIA (see Section 4.2); recovering residual

[24]In the US literature which focuses on sponsor-driven pension plans, other incentives for risk management are considered such as bankruptcy cost.

[25]If there are other assets in the pension plan, for instance because of minimum funding constraints, the insurance is bivariate, and cheaper to buy or replicate.

pension rights upon bankruptcy involves legal issues not embedded in financial instruments;[26] PIA diminishes risk-shifting incentives from the sponsorbecause as is well known, risk-based pricing in a (perpetual) insurance contract diminishes risk-shifting incentives.[27] So, protection with PIA must be cheaper than with a financial instrument.

Remark 5.1. Of course, when the traditional DB fund has (other) assets, PIA entirely solves the conflict of interest between trustees and sponsors.[28] And provided that the trustees can fulfil their duties of protection or risk management, the sponsor can make other decisions in the pension plan such as asset allocation.

Remark 5.2. With PIA, pension plans can be entirely unfunded, saving the need for a pension fund and associated expenses. An insured DB can thus resemble a (fully-insured) German book-reserved plan.[29]

[26] Insurance companies are better equipped to secure priority rights and to enforce negative pledge and similar agreements than a pension fund.

[27] This is easily understood because if the sponsor increases its leverage after $t = 0$, its insurance premiums will rise. One of the reasons why US and UK public insurance schemes were conceived as partial is precisely because at the time, researchers feared that public insurance may not be able to implement risk-based pricing, and private insurance had been discussed when the PBGC was set-up as an alternative. And even today, contributions to public insurance schemes are not fully risk-based.

[28] In fact it also solves conflicts of interests within plan members, as it is well-known that younger members have a great continuation value as employees and are ready to forego recovery contributions, whereas older members prefer securing their pension rights

[29] The PSVaG was conceived as a way of retaining public support for these schemes otherwise too risky for employees and society as a whole because pension plans without a minimum level of security are not acceptable.

6 Third-Best Non-Contractible Hybrids

Risk-shifting can be disentangled from risk-management in the pension funds' asset allocation. We focus on the risk-shifting that may arise from pension trustees: in hybrid plans, incentives are for the fund trustees to extract value from the sponsor. Then, as the value of the pension rights in the sponsor's balance sheet rises, the (value of rights from) initial funding must fall.

Problem 6.1. *The Fund's Optimisation Problem under Risk-Shifting. The fund maximises its expected utility accounting for sponsor's legal contributions. We denote $H_T = \tilde{F}_T + cp_T$ the 'total' terminal funding ratio, sum of the terminal funding ratio $\tilde{F}_T$ and of cp_T. The problem reads:*

$$\max_{\pi} \mathbb{E}_0[U(H_T)] = \max_{\pi} \mathbb{E}_0[U(\tilde{F}_T + cp_T)] \tag{6}$$

$$s.t.\ \mathbb{E}_0^{\mathbb{Q}_L}(\tilde{F}_T) = F_0 \text{ 'Asset-only budget constraint'} \tag{7}$$

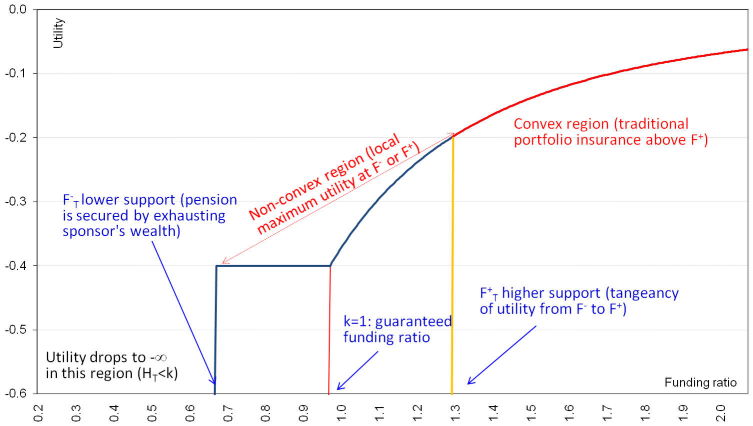
$$\text{and } cp_T = [(k - \tilde{F}_T)^+ \wedge G]$$

$$\text{with } U(H_T) = \begin{cases} -\infty & if \quad H_T < k \\ \frac{{H_T}^{1-\gamma}}{1-\gamma} & if \quad H_T \geq k \end{cases}$$

Formally, the non-contractible fund faces a non-convex problem not analysed in literature.[30] It is a mathematical extension of Carpenter (2000): the floor or lower bound of the interval in which the utility function is not convex is stochastic. Then an additional hedging demand arises against sponsor risk.

[30] Most papers assume that pension funds can only control the volatility of their investment strategy.

Figure 7: Effect of the guarantee on the utility from terminal funding ratio



The utility of the terminal ratio $\tilde{F}_T$—before the sponsor makes good on plan shortfalls—is non-convex: the utility of any $\tilde{F}_T < k$ is that of k and the utility curve is horizontal between k and F_- (the lowest funding ratio for which the sponsor can cover the shortfall with its net asset value G). As the fund has no interest in having a terminal funding ratio between F_- and F_+, it will exchange $m \cdot F^u$ against F_- (when $m \cdot F^u < F_+$).

Theorem 6.2. *Maximum Pension Risk Shifting*

The optimal terminal wealth for the fund reads (proof in Section B.2):

$$\tilde{F}_T^* = [k - G_T]^+ \cdot 1_{m \cdot F_T^u < F_T^+} + m \cdot F_T^u \cdot 1_{m \cdot F_T^u \geqslant F_T^+} \tag{8}$$

where the line $\left([k - G_T]^+, F_T^+\right)$ *is tangential to the utility curve at* F_T^+*.*[31]

This payoff can be decomposed as $\tilde{F_T} = F_T + ins_T$ with $F_T = m \cdot F_T^u \cdot 1_{m \cdot F_T^u > F_T^+}$ and $ins_T = [k - m_G \cdot G_T]^+ \cdot 1_{m \cdot F_T^u < F_T^+}$. A non-contractible fund targets $m_G = 1$ to maximise wealth extraction. This maximises the probability of being overfunded, the multiplier, and the conditional recoveries.

[31] Thus $U(F_T^+) - U(F_T^-) = (F_T^+ - F_T^-) \cdot U'(F_T^+)$ and $\frac{\gamma}{1-\gamma} \cdot {F_T^+}^{1-\gamma} - [k - G_T]^+ \cdot {F_T^+}^{-\gamma} + \frac{k^{1-\gamma}}{1-\gamma} = 0$. In general, m and π need to be calculated with simulation techniques.

7 Second-Best Outcomes with PIA

7.1 PIA as an Adequate Incentive Contract

We focus on agency costs from *pension fund* risk-shifting.[32] As the fund may maximise g by implementing theorem 6.2, the sponsor diminishes its initial funding F_0. The value of pension rights is unchanged at k' but the fund's payoff deviates from the First-Best, resulting in heavy utility losses (fig. 9).

In the principal-agent relationship considered here, the sponsor is the principal, the pension fund the agent. The hybrid pension contract is taken as given, but the principal tries to design PIA optimally so that the pension fund accepts a contract that improves the behaviour of the pension fund.

7.1.1 Definition and Pricing of the PIA Policy

The PIA contract guarantees a maximum recovery contribution c (as a percentage of the liability value) in case of sponsor failure,[33] and the protection will be offered only if no short-sales are reported by the pension fund. [34] We

[32] It is well known that risk-based pricing attenuates sponsor's risk-shifting.

[33] For PIA to act as an incentive contract (and limit risk-shifting), there must be limitations to the guarantees it provides. Note that the limitation introduced here is the exact opposite of that in public pension insurance schemes: PIA contributes to any shortfall from the first dime, but up to a limited amount c. Public pension insurance schemes, by contrast, do not guarantee first losses, but only losses beyond a threshold, and the guaranteed liability is capped. PIA acts as an incentive to pension funds controlling their shortfalls, whereas public pension insurance can lead to excessively high losses (in fact, the levy to the public pension insurance schemes can only recovered be in expectation if very high losses are possible).

[34] EU regulations facilitate verification of significant short-selling positions. The proposed E.C. (2010) is that a net aggregated short position of in a financial instrument of more than 0.2% (and each 0.1% above) of an issuer's capital outstanding is made public.

assume that the PIA policy is priced in a competitive market, *i.e.*, under $\mathbb{Q}$.

In all what follows, the sponsoring firm has initial total value G_{tot}, a pension cost of $k' = 1.3$, and the pension contract must offer a minimum guarantee $k = 1$. We assume that $\gamma = 2$. Interest rates are constant in the simulations ($L_T = 1$). After having funded F_0, the sponsoring firm invests $G_{tot} - F_0 = G_0$ in a project which has volatility $\sigma_G = 0.25$. F^u the unconstrained strategy has initial value $F_0^u = 1$ and has volatility $\sigma_F = 0.2$. The sponsor only makes a time-T recovery contributions to the pension fund, and cares about the expected value of its cash-flows.

7.2 The Optimal PIA Contract for Hybrid Plans

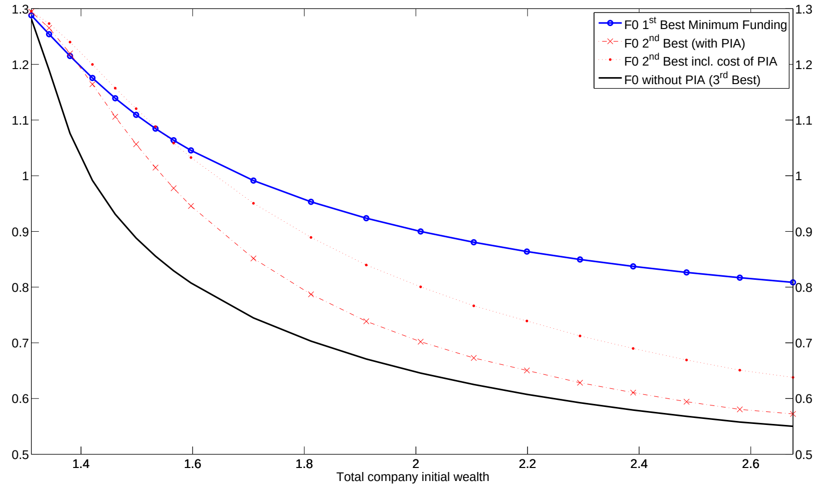
If PIA is accepted and is sufficiently valuable for the pension fund not to short the sponsor's stock, it pursues a binary strategy with budget $F_0(c) = [k - c] \cdot \mathbb{N}(d_-) + m \cdot F_0 \mathbb{N}(d_+)$. In general, PIA is expected to reduce the non-convex region and associated losses, and to improve the utility of pensioners.

Yet, the pension fund strategy remains non-convex thus Second-Best. Likewise, within the class of insurance contracts studied, the optimal PIA is that with the minimum c (if $c = 0$ was feasible, the strategy would be First-Best).[35] At the optimal c, the utility from the accepting the contract and behaving well (not shorting the sponsor's stock) is equal to the utility of the best of the alternatives (refusing the PIA contract or accepting the PIA

Esma (2011), in its summary, shows that such disclosure requirements are widespread.

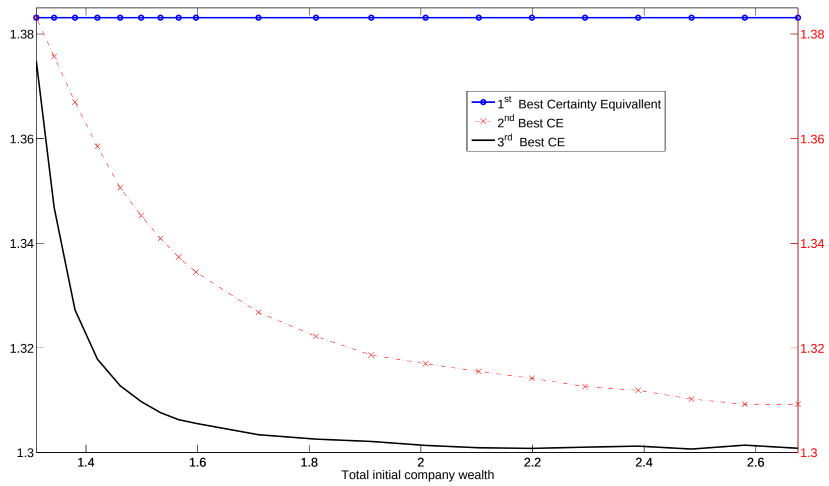
[35]By contrast, when $c = k$, the pension fund pursues $mF_T^u.1_{mF_T^u > F^+(c)}$ which maximises risk shifting, but also the utility loss.

Figure 8: Initial funding ratio with and without PIA



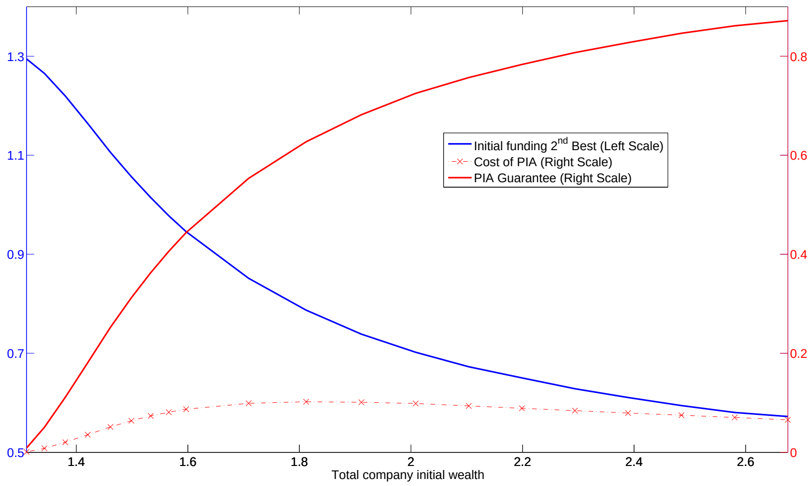
Figures 8 to 10 assume no practical difficult for shorting the sponsor's stock, and no asymmetry of information. With PIA, the initial funding ratio (red curve, and with dots including the cost of the PIA policy) is significantly above the Third-Best (black curve).

Figure 9: Certainty equivalent with and without PIA



Without PIA, as the pension payoff has a binary quality that deviates from First-Best, the richer the sponsor, the greater the utility loss – the certainty equivalent (Third-Best, black curve) converges quickly towards the cost of pensions, *i.e.*, the cost of an annuity. And PIA significantly increases the utility.

Figure 10: Optimal PIA and its cost



The required optimal (minimum) protection c increases with the sponsor size. Its cost, by contrast, is humped-shaped because there is little probability that the insurance company needs to step up and make good on pension losses from large sponsors (in relation to the size of pension liabilities).

contract and shorting the sponsor's shares – the former is always Third-Best).

Figures 8 to 10 show that without PIA, funding is low and the certainty equivallent (CE) converges towards the cost of pension, then pensioners have no more utility in a hybrid pension fund than in a traditional DB (upside is rare and brings no value). PIA increases funding, diminishes the non-convexity of the pension fund strategy, and increases the utility.

Conservative estimates[36] of the certainty equivalent of the utility gains represents the order of magnitude of the total administrative costs in pension

[36]Note that the gain studied here comes in a simple model and with inefficiencies arising solely with pension risk-shifting. Gains are greater with a lower guarantee, more complex models involving mean-reverting returns, risk-shifting from the sponsor, and asymmetry of information as analysed below.

plans: Bauer et al. (2007) report that US DB plans have average cost of 32 basis points per year and larger schemes of 20 bps.

For very large sponsors or very small pension schemes, the utility gains shrink. If one takes into account limited ability to short the sponsor's stock, the inefficiency of PIA only happens for much larger sponsors than the figures indicate.[37] The protection increases with initial total corporate wealth, but the cost of protection is humped shape.[38]

7.3 Benefits of PIA under Information Asymmetry

Implementation problems (see Section 4.2) are magnified with non-contractible hybrid plans because the optimal ins is uniquely defined with $m_G = 1$.[39] As the sponsor ignores the feasible m_G for the pension fund, a costly asymmetry of information arises, quantified assuming that the sponsor assumes that $\overline{m_G} = 1$ and $ins_T = [k - 1 \cdot G_T]^+ \cdot 1_{m_1 \cdot F_T^u < F_T^+}$ while in reality only $m_G = \frac{1}{2}$ is feasible and $ins_T = [k - \frac{1}{2} \cdot G_T]^+ \cdot 1_{m_{\frac{1}{2}} \cdot F_T^u < F_T^+}$.

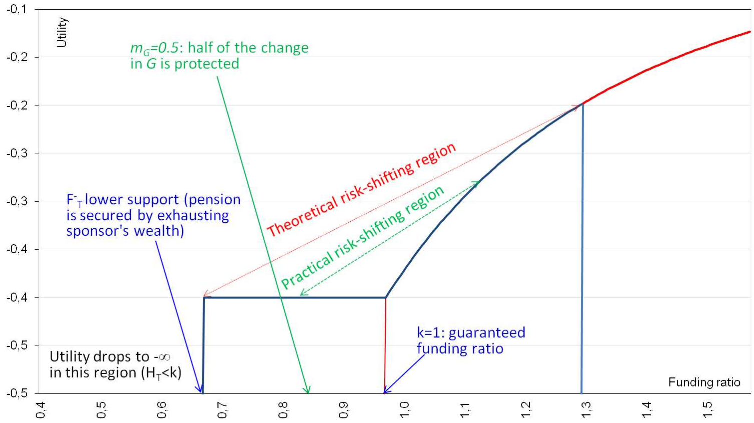
Figure 11 illustrates why losses arise, and figure 12 shows the resulting losses. Asymmetry results in excessively low initial funding. Without PIA, the fund has a certainty equivalent lower than the assumed cost of pension

[37] If $m_g = 0.25$, it is only for sponsors which net asset value is ten times the size of the pension liabilities than the benefits of PIA are shrunk. Note also that the situation where the initial value of the firm is barely sufficient to pay the pension cost, on the extreme left of the X-axis, is not relevant in practice.

[38] When G gets large the probability that $G_T < c$ when market performance is poor and that the insurance company will have to make good on a sponsor shortfall gets thinner.

[39] Not only is $m_G < 1$ required, but also eyebrows are raised when 0.5% of the market value of a firm is shorted so even shorting 50% ($m_G = 0.25$) could be difficult.

Figure 11: Utility Loss increases under information asymmetry



$m_G < 1$ shrinks the non-convex region and the risk-shifting possibility (green dotted curve) relative to that possible with $m_G = 1$ (red dotted curve). With $m_G = 0.5$ only half on the fall in G can be hedged.

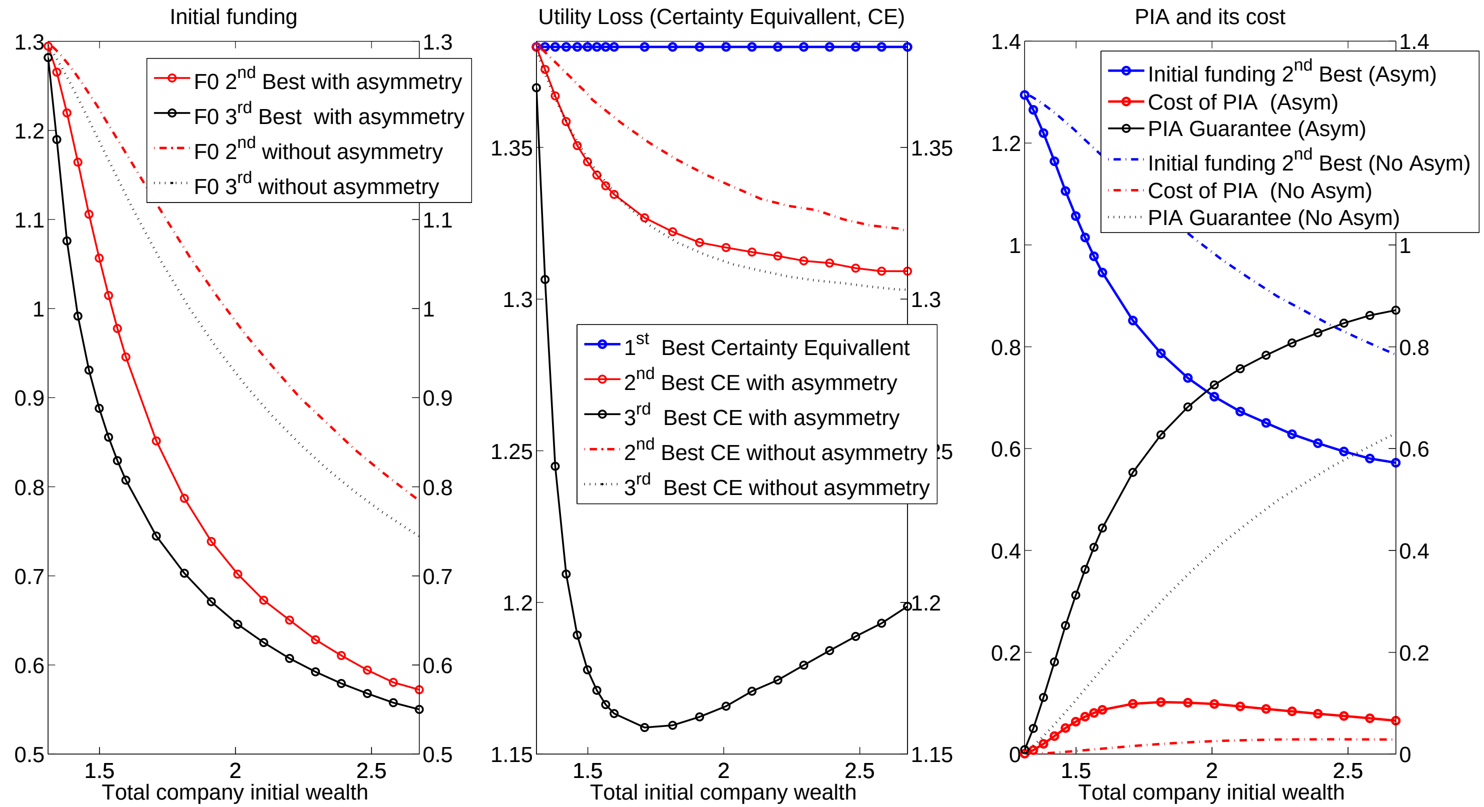
$k' = 1.3$. In fact, for some parameter values (not shown), the fund cannot even secure the pension promise (its utility is of $-\infty$): the ex-ante participation constraint is not always met unless PIA is subscribed.

PIA reduces but does not suppress the costs of information asymmetry: funding and certainty equivalent under asymmetry (plain lines) remain below full-information (dotted lines) in the two first panels.

On the whole, non-contractible pension funds lead to second-best options for plan members. It is in the interest of non-contractible pension funds to contract a PIA policy that guarantees that their risk-shifting will be limited, therefore limiting agency costs.[40]

[40] From a practical standpoint, control can be reinforced if plan trustees contract out the investment strategy to a fiduciary manager who is transparent towards the sponsor.

Figure 12: PIA policy with asymmetry of information



In these three panels, the dotted curves represent the ideal situation without asymmetry of information, and the thick curves with asymmetry of information. Asymmetry of information reduces initial funding and makes CE lesser than (assumed) cost of pension (annuity worth 1.3). PIA reduces significantly costs, improves upon third best without asymmetry. Yet costs of information asymmetry remain. If $\overline{m_g} = 0.5$, then the benefits of PIA shrink for a sponsor four or five times larger than the pension fund, and if $\overline{m_g} = 0.25$, it is only for sponsors which net asset value approaches ten times the size of the pension liabilities that the benefits of PIA are reduced.

8 Conclusion

The literature on pension fund usually ignores the risky contingent guarantees from the sponsor. Traditional portfolio insurance is impossible in underfunded funds and does not fulfill the employees ex-ante participation constraint. The First-Best (contracible) strategy requires replicating a bivariate put option on the sponsor shares. The protection of pension rights relies not on funding but on an extended class of portfolio insurance strategies.

A non-redundant asset and insurance contract, pension indemnity assurance (PIA), reduces the risk-shifting incentives from each party – fund and sponsor – and fully mitigates sponsor risk, possibly increasing the pension fund utility by an amount equal to the value of the sponsor's guarantee.

In US and UK traditional DB plans, PIA entirely solves the conflict of interest between the sponsor and its fund which trustees can then consider a sponsor's arm, leaving entirely funding and investment decision to the firm.

In European hybrid plans, under full information, improved behaviour under PIA has a utility gain equal to the costs of running a pension plan. With asymmetry of information, gains can be in order of 20% over 10 years.

Extensions are numerous. Conceptually, having disentangled risk-shifting from risk-management leads to new tests of these incentives in empirical research. Practically, risky DB pensions are inadequate incentive contracts (Drucker, 1976) for retirees cannot exert effort, but insurance makes it possible to protect retirees while keeping managers' retirement rights conditional.

A Appendices for Contractible Pension Funds

A.1 Generalised Portfolio Insurance Theorem with a Contractible Pension Fund

Problem A.1. *The Optimisation Problem in a Fair Value World*
The fund seeks to maximise the expected utility of its terminal funding, accounting for any recovery contribution in the case of underfunding:

$$\max_{\pi} \mathbb{E}_0[U(H_T)] \tag{9}$$

$$s.t.\ \mathbb{E}_0^{\mathbb{Q}_L}(F_T) = F_0 \text{ 'Asset-only budget constraint'} \tag{10}$$

$$and\ \mathbb{E}_0^{\mathbb{Q}_L}[cp_T + ins_T] = g \text{ 'Guarantee budget constraint'} \tag{11}$$

Where g is the agreed value of the guarantee and π is a self-financing strategy starting with initial wealth A_0, with: $U(H_T) = \begin{cases} -\infty & if \quad H_T < k \\ \frac{{H_T}^{1-\gamma}}{1-\gamma} & if \quad H_T \geq k \end{cases}$

Lemma A.2. *Characterisation of the downside and of insurance.*
For any feasible strategy, $\mathbb{E}_0^{\mathbb{Q}_L}[cp_T + ins_T] = g$ is equivalent to having both $\mathbb{E}_0^{\mathbb{Q}_L}[[k - F_T]^+] = g$ and $\forall F_T,\ cp_T + ins_T = [k - F_T]^+$.

Proof.

It is straightforward to show that:
$\mathbb{E}_0^{\mathbb{Q}_L}[[k - F_T]^+] < g$ implies $\mathbb{E}_0^{\mathbb{Q}_L}[cp_T + ins_T] < g$
$\mathbb{E}_0^{\mathbb{Q}_L}[[k - F_T]^+] > g$ implies that the strategy is not feasible, either because

$\mathbb{E}_0^{\mathbb{Q}_L}[cp_T + ins_T] > g$ and the fair value budget constraint in (2) is not respected, or because there is terminal underfunding with a positive probability and expected utility is of $-\infty$. Then $U(H_T) = F_T \vee k$. These results can be viewed as a simple reinterpretation of the assumptions. □

Remark A.3. The characterisation of the downside in Lemma A.2 allows the problem to be rewritten in a much simpler way:

$$\begin{cases} \max\limits_{\pi} \mathbb{E}_0[U(H_T)] = \max\limits_{\pi} \mathbb{E}_0[U(F_T \vee k)] & \\ \text{s.t. } \mathbb{E}_0^{\mathbb{Q}_L}(F_T) = F_0 & \text{`Asset-only budget constraint'} \\ \text{and } \mathbb{E}_0^{\mathbb{Q}_L}[k - F_T]^+ = g & \text{`Guarantee budget constraint'} \\ \text{where g is the agreed guarantee value} & \text{(and } \pi \text{ a self-financing strategy)} \end{cases} \tag{12}$$

The 'Ex-ante participation constraint' simply reads $cp_T + ins_T = [k - F_T]^+$.

Theorem A.4. *Generalised portfolio insurance theorem*
F_T^* denotes the optimal strategy or terminal payoff. In the region $F_T^* > k$, F_T^* is equal to F_T^{pref}. In the region where $F_T^* \leq k$, the pension fund is indifferent to the value of F_T^* as long as the guarantee budget constraint holds.

$$H_T^* = F_T^{pref} = \begin{cases} m^* \cdot F_T^u & \text{if} \quad m^* \cdot F_T^u > k \\ k & \text{if} \quad m^* \cdot F_T^u \leqslant k \end{cases} \tag{13}$$

$$F_T^* = \begin{cases} H_T^* = F_T^{pref} = m^* \cdot F_T^u & \text{if} \quad m^* \cdot F_T^u > k \\ \text{Any payoff } F_T \leqslant k & \text{if} \quad m^* \cdot F_T^u \leqslant k \\ \qquad s.t. \quad \mathbb{E}_0^{\mathbb{Q}_L}[k - F_T]^+ = g & \end{cases}$$

A.2 Proof (Contractible Pension Fund)

Definition A.5. Feasible strategies.

A feasible strategy H_T has an expected utility higher than $-\infty$.

Lemma A.6. *(Reminder: optimal constrained strategy for pre-funded 'DCs') The optimal strategy for the pre-funded plan is uniquely defined as* $[m^* \cdot F_T^u \vee k]$ *where* $F_T^u = (\upsilon_0 \cdot Z_T \cdot L_T)^{-\frac{1}{\gamma}}$ *is the optimal strategy for the unconstrained investor (*υ_0*, a positive number, ensures the respect of the budget constraint).*

Proof. of Lemma. The optimal pre-funded strategy F_T^{pref} with initial funding ratio $F_0 + g = H_0$ and no further support from the sponsor can be solved with the martingale approach of Cox and Huang (1989).[41] This intermediate proof (which abstracts from the sponsor's guarantee) is a standard textbook derivation, so the reader may skip to the proof of theorem A.2 on page 37.

- We first solve the **unconstrained** static maximisation program:

[41] See Cvitanic and Zapatero (2004) for more modern notations and Martellini and Milhau (2009) for a similar example with liabilities. In complete markets, the duality or martingale approach used is equivalent to solving the Hamilton-Jacobi-Bellman partial differential equation, yet it is much more simple because it relies on static optimisation of the optimal terminal payoff. In a second step, the pricing of the optimal payoff allows for dynamic replication and definition of the self-financing strategy π.

$$\max_{\pi} E_0\left[\frac{(A_T/L_T)^{1-\gamma}}{1-\gamma}\right]$$
$$s.t.\ A_0 = \mathbb{E}_0^{\mathbb{Q}}[\frac{A_T}{e^{rT}}] = \mathbb{E}_0[Z_T \cdot A_T] = L_0 \cdot \mathbb{E}_0^{\mathbb{Q}_L}[\frac{A_T}{L_T}] \Longleftrightarrow \mathbb{E}_0^{\mathbb{Q}_L}(F_T) = F_0$$

Given the states of the world, the first-order condition reads $\frac{1}{L_T}\left(A_T^u / L_T\right)^{-\gamma} = \upsilon_0 \cdot Z_T$. So, $A_T^u = \upsilon_0{}^{-1/\gamma} \cdot Z_T{}^{-\frac{1}{\gamma}} \cdot L_T^{1-\frac{1}{\gamma}}$, and $F_T^u = \upsilon_0{}^{-\frac{1}{\gamma}} \cdot (L_T Z_T)^{-\frac{1}{\gamma}}$

- We then solve for the **constrained** program:

$$\max_{\pi} \mathbb{E}_0\left[\frac{(A/L)^{1-\gamma}}{1-\gamma}\right] \quad \text{s.t. } A_0 = \mathbb{E}_0^{\mathbb{Q}}[Z_T \cdot A_T] \quad \text{and } F_T \geqslant k$$

The first-order conditions read: $\frac{1}{L_T} \cdot F_T^{pref^{-\gamma}} - \upsilon_k \cdot Z_T + \frac{\upsilon_c(k)}{L_T} = 0$

where υ_0 and υ_k are the Lagrange multipliers associated with the budget constraint (for $k = 0$ or k), $\upsilon_c(k)$ with the minimum funding constraint k.

- $\upsilon_c(k) = 0$ when $F_T^{pref} > k$, and is otherwise binding, so we have:

$$F_T^{pref} = \begin{cases} \upsilon_k^{-\frac{1}{\gamma}} \cdot (Z_T \cdot L_T)^{-\frac{1}{\gamma}} & \text{if } Z_T \cdot L_T < \underline{ZL} = \frac{k^{-\gamma}}{\upsilon_k} \\ k & \text{if } \underline{ZL} < Z_T \cdot L_T \end{cases} \tag{14}$$

Or:

$$F_T^{pref} = \begin{cases} m^* \cdot F_T^u & \text{if } m^* \cdot F_T^u > k \\ k & \text{if } m^* \cdot F_T^u \leqslant k \end{cases} \tag{15}$$

$\underline{ZL}$ represents the threshold $m^* \cdot F_T^u = k$ and $m = \left(\frac{\upsilon_k}{\upsilon_0}\right)^{-\frac{1}{\gamma}}$.

- Finally, the participation rate m must respect the pre-funded strategy budget constraint $F_0 + g$.[42] When the parameters set is constant, assets, liabilities and the *s.d.f.* are log-normals, and so is $Z^{-\frac{1}{\gamma}}$ and its product by

[42] m is defined in $t = 0$ and the pension fund commits to the $t = 0$ strategy; this is by contrast to the program in Appendix B, where a time-t optimisation would yield the same results and the same m than at $t = 0$.

L_T, F^u_T. Noting that $\mathbb{E}^{\mathbb{Q}_L}[k \vee m^* \cdot F^u_T] = k + \mathbb{E}^{\mathbb{Q}_L}[F^u_T - k]^+$ and making use of the Black-Scholes formula implies:

$$F_0 + g = m^* \cdot F_0 \cdot \mathbb{N}(d_+(m^* \cdot F_0, k)) + k \cdot \mathbb{N}(-d_-(m^* \cdot F_0, k)) \tag{16}$$

with $d\pm = \frac{log(m^* \cdot F_0/k) \pm \frac{1}{2}\sigma^2_{F^u} T)}{\sigma_{F^u}\sqrt{T}}$ and $\sigma_{F^u} = \frac{||\theta' - \sigma_L||}{\gamma}$. The volatility of F^u is the difference between σ_{A^u} and σ_L; A^u has exposure $\frac{1}{\gamma}$ to a portfolio with volatility θ' and $1 - \frac{1}{\gamma}$ to a portfolio with volatility that of liabilities (with minor notation abuse between vector and scalar volatility).[43] □

Proof. Proof of theorem A.4.

This proof relies on the 'verification theorem': we first verify that the terminal payoff H_T is that of the pre-funded strategy then that both the 'asset-only' $\mathbb{E}^{\mathbb{Q}_L}_0(F_T) = F_0$ and 'guarantee' (in A.2) budget constraints are satisfied.

- When $m^* \cdot F^u_T \leqslant k$, the total terminal payoff $H_T = k$ from A.2, and when $m^* \cdot F^u_T > k$, $H_T = F^{pref}_T$, so these payoffs are always equal.

- By construction, A.2 respects the guarantee budget constraint as when $m^* \cdot F^u_T \leqslant k$, $\mathbb{E}^{Q_L}_0[F_T \cdot \mathbb{1}_{F_T \leqslant k}] = \mathbb{E}^{Q_L}_0[(k - [k - F_T])\mathbb{1}_{m^* \cdot F^u_T \leqslant k}] = k \cdot N(-d_-) - g$.[44]

[43] A more general and elegant proof involves using A^u as a numeraire because F^u has drift $+\sigma^2_{F^u}$ under $\mathbb{Q}_{A^u}$, so $log(F^u_T)$ is a normal with mean $+\frac{1}{2}\sigma^2_{F^u}T$. After all, changing numeraire from L to A requires an adjustment with $\sigma_{A^u} - \sigma_L = \sigma_{F^u}$, and it is the equivalent when under $\mathbb{Q}_L$ to use a F^u as a (relative) numeraire. Then, the budget constraint for m reads: $\mathbb{E}^{\mathbb{Q}_L}[k \vee m^* \cdot F^u_T] = m^* \cdot F_0 \cdot \mathbb{E}^{\mathbb{Q}_A}[\mathbb{1}_{F^u_T > k/m}] + k \cdot \mathbb{E}^{\mathbb{Q}_L}[\mathbb{1}_{F^u_T < k/m}]$.

[44] One can verify that with constant parameter set,

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}_L}_0[F_T] &= \mathbb{E}^{\mathbb{Q}_L}_0[F_T \cdot \mathbb{1}_{m^* \cdot F^u_T \leqslant k}] + \mathbb{E}^{\mathbb{Q}_L}_0[F_T \cdot \mathbb{1}_{m^* \cdot F^u_T > k}] \\
&= \mathbb{E}^{\mathbb{Q}_L}_0[F_T \cdot \mathbb{1}_{m^* \cdot F^u_T \leqslant k}] + \mathbb{E}^{\mathbb{Q}_L}_0[F^{pref}_T \cdot \mathbb{1}_{m^* \cdot F^u_T > k}] \\
&= k \cdot \mathbb{N}(-d_-) - g + m^* \cdot F^u_T \cdot \mathbb{N}(d_+) \\
&= (F_0 + g) - g = F_0
\end{aligned}$$

Of course, one can also explicitly verify that any strategy that leads to a different total terminal payoff has lower expected utility. □

A.3 Stochastic Floor Representation of the Strategy

An optimal strategy can be characterised in two ways:

- Theorem 3.1 relies on an explicit separation between the underlying portfolio insurance strategy and its protection against sponsor risk.

- An optimal generalised portfolio insurance strategy can be alternatively characterised as a portfolio insurance strategy with a stochastic floor – the option to exchange $m^* \cdot F^u$ against $\tilde{k} = [k \wedge [k - m_G \cdot G_T + \delta]^+]$ is an optimal strategy:[45] $\tilde{F}_T^* = \tilde{k} + [m^* \cdot F^u - \tilde{k}]^+$ (and $\tilde{F}_T^* = F_T^* + ins_T$.) (17)

And one can check that $[k - \tilde{F}_T^*]^+ \leqslant m_G \cdot G_T < G_T$, so the sponsor can always cover deficits and H_T in (17) is equal to that in theorem 3.1 on page 12. Both have the same participation rate and their payoff in the region where $F_T^* > k$ is equal to $m^* \cdot F_T^u | m^* \cdot F_T^u > k$: the two ways of exposing the problem are strictly equivalent.[46] However, disentangling insurance makes the solution to the problem very easy to understand, to solve and to implement.

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[45] The class of optimal strategies can be proved to be characterised as portfolio insurance with variable floor; proof is omitted in short versions of the paper.

[46] And, of course, since the replication of this strategy would require an exposure to both F_T^u and to $\tilde{k} = [(k - m_G \cdot G_T) \wedge \underline{k}]$, a negative exposure to G is necessary, implying a hedging demand against the risk of a fall in the sponsor's net asset value.

B Appendix for Non-Contractible Pension Funds

Problem B.1. *The fund's problem is to maximise its expected utility:*

$$\max_{\pi} \mathbb{E}_0[U(H_T)] = \max_{\pi} \mathbb{E}_0[U(\tilde{F}_T + cp_T)] \tag{18}$$

$$s.t.\ \mathbb{E}_0^{\mathbb{Q}_L}(\tilde{F}_T) = F_0 \text{ 'Asset-only budget constraint'} \tag{19}$$

$$\text{with } U(H_T) = \begin{cases} -\infty & if \quad H_T < k \\ \frac{{H_T}^{1-\gamma}}{1-\gamma} & if \quad H_T \geq k \end{cases}$$

Theorem B.2. *Generalised Pension Risk-Shifting Portfolio Insurance*

$\tilde{F}_T^*$ *denotes the optimal strategy or terminal payoff.* $F_T^- = [k - G_T]^+$ *denotes the low reference funding ratio, and* F_T^+ *is the high-reference funding ratio, where* F_T^+ *solves* $\frac{\gamma}{1-\gamma} \cdot {F_T^+}^{1-\gamma} - [k - G_T]^+ \cdot {F_T^+}^{-\gamma} + \frac{k^{1-\gamma}}{1-\gamma} = 0$

$$H_T^* = \begin{cases} m \cdot F_T^u & \text{if} \quad m \cdot F_T^u > F_T^+ \\ k & \text{if} \quad m \cdot F_T^u \leqslant F_T^+ \end{cases} \tag{20}$$

$$\text{and } \tilde{F}_T^* = \begin{cases} H_T^* = m \cdot F_T^u & \text{if} \quad m \cdot F_T^u > F_T^+ \\ F_T^- = [k - G_T]^+ & \text{if} \quad m \cdot F_T^u \leqslant F_T^+ \end{cases}$$

Proof. Proof of theorem B.2.

The duality approach can be used in non-convex maximisation, see Basak and

Shapiro (2001) or Carpenter (2000). The pension fund solves the program:

$$\max_{\pi}\mathbb{E}_0\left[\frac{(\tilde{F}_T+[[k-\tilde{F}_T]^+\wedge G_T])^{1-\gamma}}{1-\gamma}\right]$$

$$\text{s.t. } A_0=\mathbb{E}_0[Z_T\cdot A_T] \text{ `Budget constraint', and}$$

$$\forall\tilde{F}_T<k,[k-\tilde{F}_T]^+\leqslant G_T \qquad \text{`Feasibility constraint (finite utility)'} \quad (21)$$

Supposing positivity constraints on the terminal wealth, from the feasibility constraint in (21), we must have $\tilde{F}_T\geqslant F_T^-=[k-G_T]^+$ *a.s.* to ensure $H_T\geqslant k$. With similar arguments to those used to state problem A.1, as the utility of $\tilde{F}_T\in\,][k-G_T]^+,k[$ is constant, the first-order conditions read:

$$\begin{cases}\frac{1}{L_T}\cdot\tilde{F}_T^{*-\gamma}-\upsilon\cdot Z_T-\upsilon_G\cdot\frac{Z_T}{L_T}=0 & \text{if } \tilde{F}_T^*\geqslant k \text{ or } [k-\tilde{F}_T]^+=G\\ \text{any value } \tilde{F}_T{}^*\geqslant[k-G_T]^+ & \text{if } \tilde{F}_T<k\end{cases} \quad (22)$$

With υ and υ_G the Lagrange multipliers associated with the budget and feasibility constraints, and $\upsilon_G=0$ if $\tilde{F}_T^*\geqslant k$, so that we can rewrite (22) as:

$$\tilde{F}_T^*=m\cdot F_T^u \qquad \text{if } \tilde{F}_T^*\geqslant k \quad (23)$$

$$\text{any value } \tilde{F}_T{}^*\geqslant[k-G_T]^+ \qquad \text{if } \tilde{F}_T<k \quad (24)$$

But the function of $\tilde{F}_T$ over which $max\cdot$ operates in (21) is not concave between F_T^- and F_T^+, so that in the interval $[F_T^-,F_T^+]$, $\tilde{F}_T{}^*$ must be equal to either of these values. As a consequence, after substitution for $F_T^u=$

$\upsilon_0^{-\frac{1}{\gamma}} \cdot (Z_T \cdot L_T)^{-\frac{1}{\gamma}}$, we have with $m = \left(\frac{\upsilon}{\upsilon_0}\right)^{-\frac{1}{\gamma}}$:

$$\tilde{F}_T^* = \begin{cases} m \cdot F_T^u & \text{if} \quad m \cdot F_T^u \geqslant F_T^+ \\ F_T^- = [k - G_T]^+ & \text{if} \quad m \cdot F_T^u < F_T^+ \end{cases} \tag{25}$$

One can formally check that in (23) for $\tilde{F}_T^* < k$, $\tilde{F}_T^* = F_T^-$ because $\tilde{F}_T^* = F_T^+ \geqslant k$ is not feasible at the same time that $\tilde{F}_T^* < k$. Likewise, in (24) for $\tilde{F}_T^* > k$, one must either have $\tilde{F}_T^* = m \cdot F_T^u$ and $\tilde{F}_T^* \geqslant F_T^+$ or $\tilde{F}_T^* = F_T^- \leqslant k$, and, as the later is not feasible, we must have $\tilde{F}_T^* = m \cdot F_T^u$ (only) when $m \cdot F_T^u \geqslant F_T^+$.[47]

□

[47] One could also directly use the derivation in Carpenter (2000), taking liabilities into account, replacing the zero in the all-or-nothing payoff by $F_T^- = [k - G_T]^+$ (and subsequently translating payoffs greater than F_T^+).

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