Properties of the Cepstrum

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September 29, 2006

1. Definition of the Cepstrum that we shall use

There are many definitions of the cepstrum. For the purposes of this note, we will assume a signal $x(t)$ sampled at every $T=1/N$ seconds, for an integer N. We will let $F(x)$ denote the N-point discrete Fourier Transform of x. So, if $j=\sqrt{-1}$,

$$F(x)_k = \sum_{p=0}^{N-1} x(p\,T)\exp\left(\frac{2\,\pi\,j\,p\,k}{N}\right), \text{ for } 0 \leqslant k \leqslant N-1 \tag{1}$$

The inverse discrete Fourier Transform will be denoted $G(x)$ and is defined by the formula

$$G(x)_k = \frac{1}{N}\sum_{p=0}^{N-1} F(x)_p \exp\left(-\frac{2\,\pi\,j\,p\,k}{N}\right), \text{ for } 0 \leqslant k \leqslant N-1 \tag{2}$$

This follows the Matlab convention (Ref 2). Mathcad (Ref 3) follows a different convention, defining its function cfft by the formula

$$F(x)_k = \frac{1}{\sqrt{N}}\sum_{p=0}^{N-1} x(p)\exp\left(\frac{2\,\pi\,j\,p\,k}{N}\right)$$

and similarly with its icfft.

Note 1. The Fourier Transform should satisfy the property that if one applies it 4 times, one winds up with the original sequence, i.e. $F(F(F(F(x))))=x$. This is true for the Mathcad definition but not the Matlab one. I believe most authors use the Matlab convention.

The cepstrum of x, $C(x)$, is defined as follows, following the definition of the Matlab function **rceps**.

Definition 2. *For a complex number* $z = x + j\,y$, *let* real(z) *denote the real part of* z *(i.e.* x*) and* abs(z) *denote the modulus of the complex number, i.e.* $\sqrt{x^2+y^2}$.

Definition 3. *(cepstrum) If log denotes logarithm to the base* e *(the Matlab convention),*

$$C(x)_k = \mathrm{real}(G(\log(\mathrm{abs}(F(x)_k))),\ \ 0 \leqslant k \leqslant N-1 \tag{3}$$

Note 4. There are many definitions of the cepstrum in the literature. The Wikipedia article on the cepstrum states that the inverse DFT was not in the Bogart's original definition of the cepstrum (Ref. 1), but its use is widespread.

The N-point DFT of $\sin(2\ \pi\ p\ t)$ for a positive integer $p<N/2$ is $\frac{N}{2\,j}$ $(\delta_{p,N-n} - \delta_{p,n})$, where $\delta_{p,n}$ denotes the Kronecker delta, i.e

$$\delta_{p,n} = \begin{cases} 1 \text{ if } p = n \\ 0 \text{ if } p \neq n \end{cases}$$

The cepstrum is therefore not defined for integer frequencies because all but two entries of the DFT are 0, and therefore have a log of $-\infty$.

2. The cepstrum and convolutions

Let $x \star y$ denotes the convolution of the sequences x and y. Then it is easy to show that

Proposition 5. $C(x * y) = C(x) + C(y)$

Proof. We begin by noting that, for $0 \leqslant k \leqslant N-1$, $F(x \star y)_k = F(x)_k F(y)_k$ and so

$$\mathrm{abs}(F(x * y)_k) = \mathrm{abs}(F(x)_k) \times \mathrm{abs}(F(y)_k)$$

The next stage in forming the cepstrum is to take logs, so

$$\log(\mathrm{abs}(F(x * y)_k)) = \log(\mathrm{abs}(F(x)_k)) + \log(\mathrm{abs}(F(y)_k))$$

As the functions real and G are linear, equation (4) follows. □

The following may be of interest. The proof is omitted.

Lemma 6. *For any complex constant* λ*, independent of k,* $C(\lambda x)_k = C(x)_k$ *when* $k > 0$*.*

The significance is that the DC value of the cepstrum is ignored, and what this says is that multiplication of the function $x(t)$ by a constant will only affect the DC term and is irrelevant to the cepstrum processing.

3. DFT of $\exp(2\pi j f t)$, where f is not an integer

The objective is to prove that this is

$$F(x)_k = \exp\left[\pi j \left(f - \frac{f+k}{N}\right)\right] \frac{\sin(\pi f)}{\sin(\pi (f+k)/N)} \tag{4}$$

This is of interest because we want to find the DFT of $\sin(2\pi f t) = \frac{\exp(2\pi j f t) - \exp(-2\pi j f t)}{2j}$. Hence equation (4) will help us find what we want.

We note the following

Lemma 7. *For any complex number z,* $\sum_{p=0}^{N-1} z^p = (1 - z^N)/(1 - z)$

The Fourier Transform of $\exp(2\pi j f t)$ is (from equation (1), as $\exp(2\pi j k) = 1$),

$$F(x)_k = \sum_{p=0}^{N-1} \exp\left(\frac{2\pi j f p}{N}\right) \exp\left(\frac{2\pi j p k}{N}\right) = \sum_{p=0}^{N-1} \left[\exp\left(\frac{2\pi j f}{N}\right) \exp\left(\frac{2\pi j k}{N}\right)\right]^p$$

and so, by Lemma 7,

$$F(x)_k = \frac{1-\exp(2\,\pi\,j\,f)}{1-\exp\left(\frac{2\,\pi\,j\,f}{N}\right)\exp\left(\frac{2\,\pi\,j\,k}{N}\right)} = \frac{1-\exp(2\,\pi\,j\,f)}{1-\exp\left(\frac{2\,\pi\,j\,(f+k)}{N}\right)} \tag{5}$$

The next step in simplifying (5) is to note the following easily proved fact

Lemma 8. *For real numbers α and β,*

$$\frac{1-\exp(j\,\alpha)}{1-\exp(j\,\beta)} = \exp\left(j\,\frac{\alpha-\beta}{2}\right)\frac{\sin(\alpha/2)}{\sin(\beta/2)}$$

Proof. We note that $1-\exp(j\,\alpha) = \exp(j\,\alpha/2)\,(\exp(-\,j\,\alpha/2) - \exp(j\,\alpha/2))$. Recalling Euler's theorem $\exp(j\theta) = \cos(\theta) + j\sin(\theta)$, we deduce that $1-\exp(j\,\alpha) = -\,2\,j\exp(j\,\alpha/2)\sin(\alpha/2)$. The result immediately follows □

We deduce from equation (5) and Lemma 8 that

Proposition 9. *The N point DFT $F(x)_k$ of $\exp(2\,\pi\,j\,f\,t)$ is given by the formula*

$$F(x)_k = \exp\left[\pi\,j\left(f-\frac{f+k}{N}\right)\right]\frac{\sin(\pi\,f)}{\sin(\pi\,(f+k)/N)}$$

This formula only applies when f is not an integer.

4. DFT of $\sin(2\,\pi\,f\,t)$, where f is not an integer

We begin by noting the formula

$$\sin(2\,\pi\,f\,t) = \frac{1}{2\,j}\,(\exp(2\,\pi\,j\,f\,t) - \exp(-\,2\,\pi\,j\,f\,t)) \tag{6}$$

Using Proposition 9, we deduce that the DFT is

$$\frac{1}{2\,j}\exp\left[\pi\,j\left(f-\frac{f+k}{N}\right)\right]\frac{\sin(\pi\,f)}{\sin(\pi\,(f+k)/N)} - \frac{1}{2\,j}\exp\left[-\,\pi\,j\left(f-\frac{f-k}{N}\right)\right]\frac{\sin(\pi\,f)}{\sin(\pi\,(f-k)/N)}$$

This can be reduced to the following

$$F(x)_k = \frac{1}{2\,j}\exp\left[-\,\pi\,j\left(f-\frac{f-k}{N}\right)\right]\sin(\pi\,f)\left\{\frac{\exp\left[2\,\pi\,j\left(f-\frac{f}{N}\right)\right]}{\sin(\pi\,(f+k)/N)} - \frac{1}{\sin(\pi\,(f-k)/N)}\right\}$$

The expression $\sin(\pi\,f)/(2\,j)$ is independent of k. Let

$$G(x)_k = \exp\left[-\,\pi\,j\left(f-\frac{f-k}{N}\right)\right]\left\{\frac{\exp\left[2\,\pi\,j\left(f-\frac{f}{N}\right)\right]}{\sin(\pi\,(f+k)/N)} - \frac{1}{\sin(\pi\,(f-k)/N)}\right\}$$

So that

$$F(x)_k = \frac{1}{2\,j}\sin(\pi\,f)\,G(x)_k \tag{7}$$

Then (using Maxima - Ref 4 - to simplify the expressions),

$$\text{real}(G(x)_k) = \frac{\sin\left(\frac{2\,\pi\,k}{N} + \pi\,f\right) + \sin\left(\frac{2\,\pi\,k}{N} - \pi\,f\right)}{\cos\left(\frac{2\,\pi\,f}{N}\right) - \cos\left(\frac{2\,\pi\,k}{N}\right)} \tag{8}$$

$$\text{imag}(G(x)_k) = \frac{\cos\left(\frac{2\,\pi\,k}{N} + \pi\,f\right) + \cos\left(\frac{2\,\pi\,k}{N} - \pi\,f\right) - 2\cos\left[\pi\,f\left(1 - \frac{2}{N}\right)\right]}{\cos\left(\frac{2\,\pi\,f}{N}\right) - \cos\left(\frac{2\,\pi\,k}{N}\right)} \tag{9}$$

Beginning with the numerators in equations (8) and (9), we can apply trigonometric identities to find that

$$\sin\left(\frac{2\,\pi\,k}{N} + \pi\,f\right) + \sin\left(\frac{2\,\pi\,k}{N} - \pi\,f\right) = 2\sin\left(\frac{2\,\pi\,k}{N}\right)\cos(f\,\pi)$$

$\cos\left(\frac{2\,\pi\,k}{N} + \pi\,f\right) + \cos\left(\frac{2\,\pi\,k}{N} - \pi\,d\right) - 2\cos\left[\pi\,f\left(1 - \frac{2}{N}\right)\right] = 2\cos\left(\frac{2\,\pi\,k}{N}\right)\cos(f\,\pi) - 2\cos\left[\pi\,p\left(1 - \frac{2}{N}\right)\right]$

Thus by equation (7), (8) and (9), as $\sin(2\,\pi\,f) = 2\sin(\pi\,f)\,\cos(\pi\,f)$, we conclude that

Proposition 10. *The N point DFT of* $\sin(2\,\pi\,f\,t)$ *is (when f is not an integer) is given by the following formula which applies for* $0 \leqslant k \leqslant N-1$,

$$F(x)_k = \frac{1}{2}\,\frac{\sin(2\,\pi\,f)}{\cos\left(\frac{2\,\pi\,f}{N}\right) - \cos\left(\frac{2\,\pi\,k}{N}\right)}\left(\exp\left(-\frac{2\,\pi\,j\,k}{N}\right) - \frac{\cos\left[\pi\,f\left(1 - \frac{2}{N}\right)\right]}{\cos(\pi\,f)}\right) \tag{10}$$

5. The case f is close to being an integer

We will suppose that $f = p + \varepsilon$, where p denotes a positive integer less than $N/2$ and ε is small, so that ε^2 is negligible.

Observe that $\sin(2\,\pi\,(p+\varepsilon)) = \sin(2\,\pi\,p + 2\,\pi\,\varepsilon) = \sin(2\,\pi\,\varepsilon) = 2\,\pi\,\varepsilon$, to first order in ε. We deduce from equation (10) that, when $k \neq p$ and $k \neq N - p$,

$$F(x)_k = \frac{1}{2}\,\frac{2\,\pi\,\varepsilon}{\cos\left(\frac{2\,\pi\,p}{N}\right) - \cos\left(\frac{2\,\pi\,k}{N}\right) + O(\varepsilon)}\left(\exp\left(-\frac{2\,\pi\,j\,k}{N}\right) - \frac{\cos\left[\pi\,p\left(1 - \frac{2}{N}\right)\right]}{\cos(\pi\,p)} + O(\varepsilon)\right)$$

We conclude that

$$F(x)_k = \frac{\pi\,\varepsilon}{\cos\left(\frac{2\,\pi\,p}{N}\right) - \cos\left(\frac{2\,\pi\,k}{N}\right)} \left(\exp\left(-\frac{2\,\pi\,j\,k}{N} \right) - \frac{\cos\left[\pi\,p\left(1 - \frac{2}{N}\right)\right]}{\cos(\pi\,p)} \right) + O(\varepsilon^2) \qquad (11)$$

This can be simplified down to

$$F(x)_k = -\pi\,\varepsilon + j\,\frac{\pi\,\varepsilon\sin(2\,\pi\,k/N)}{\cos(2\,\pi\,p/N) - \cos(2\,\pi\,k/N)} + O(\varepsilon^2)$$

This concludes the case $k \neq p,\, N - p$. The case $k = p$ will now be considered. Both the numerator and denominator of $F(x)_k$ are $O(\varepsilon)$, so we need to go to $O(\varepsilon^2)$.

Using Mathcad 13's Maple package, the numerator in equation (10), viz.

$$\sin(2\,\pi\,f) \left(\exp\left(-\frac{2\,\pi\,j\,k}{N} \right) - \frac{\cos\left[\pi\,f\left(1 - \frac{2}{N}\right)\right]}{\cos(\pi\,f)} \right)$$

can be reduced to

$$-2\,(\pi\,\varepsilon)\,j\sin\left(2\,\pi\frac{p}{N}\right) - 2\,(\pi\,\varepsilon)^2\sin\left(2\,\pi\frac{p}{N}\right)\left(1 - \frac{2}{N}\right)$$

The denominator in equation (10), viz $\cos\left(\frac{2\,\pi\,f}{N}\right) - \cos\left(\frac{2\,\pi\,k}{N}\right)$, reduces to

$$-\frac{2\,(\pi\,\varepsilon)}{N}\sin\left(\frac{2\,\pi\,p}{N}\right) - \frac{2\,(\pi\,\varepsilon)^2}{N^2}\cos\left(\frac{2\,\pi\,p}{N}\right) + O(\varepsilon^3)$$

Applying equation (10), we conclude that, up to $O(\varepsilon)$,

$$F(x)_p = N\,j/2 + (\pi/2)\,(N-2)\,\varepsilon - j\,(\pi/2)\,\cot(2\,\pi\,p/N)\,\varepsilon$$

This concludes the case $k = p$. The above argument assumes that $\sin\left(\frac{2\,\pi\,p}{N}\right) \neq 0$.

In the case $k = N - p$, Maxima was used to establish that the numerator becomes

$$2\,(\pi\,\varepsilon)\,j\sin\left(2\,\pi\frac{p}{N}\right) - 2\,(\pi\,\varepsilon)^2\sin\left(2\,\pi\frac{p}{N}\right)\left(1 - \frac{2}{N}\right) + O(\varepsilon^3).$$

The denominator remains the same. Putting this all together, we have shown that

Proposition 11. *When* $f = p + \varepsilon$, *for integer* p *and small* ε, *and* $\sin(2\,\pi\,p/N) \neq 0$, *then the* N*-point DFT of* $\sin(2\,\pi\,f\,t)$ *is, to first order in* ε, *for* $0 \leqslant k \leqslant N-1$,

$$F(x)_k = \begin{cases} N\,j/2 + (\pi/2)\,(N-2)\,\varepsilon - j\,(\pi/2)\,\cot(2\,\pi\,p/N)\,\varepsilon, \text{ when } k = p \\ -N\,j/2 + (\pi/2)\,(N-2)\,\varepsilon + j\,(\pi/2)\,\cot(2\,\pi\,p/N)\,\varepsilon, \text{ when } k = N - p \\ -\pi\,\varepsilon + j\,\dfrac{\pi\,\varepsilon\sin(2\,\pi\,k/N)}{\cos(2\,\pi\,p/N) - \cos(2\,\pi\,k/N)}. \text{ when } k \neq p, N - p \end{cases}$$

6. The cepstrum when $f = p + \varepsilon$

We apply Proposition 11 to find the cepstrum. Firstly we need to find $\mathrm{abs}(F(x)_k)$ to $O(\varepsilon)$. This is easily done and the result is as follows

$$\mathrm{abs}(F(x)_k) = \begin{cases} N/2 - (\pi/2)\cot(2\pi p/N)\,\varepsilon, \text{ when } k = p, N-p \\ \pi\,\varepsilon\sqrt{1 + \left(\dfrac{\sin(2\pi k/N)}{\cos(2\pi p/N) - \cos(2\pi k/N)}\right)^2}\,.\text{ when } k \neq p, N-p \end{cases}$$

Thus, to $O(\varepsilon)$,

$$\log(\mathrm{abs}(F(x)_k)) = \begin{cases} \log(N/2) - (\pi/N)\cot(2\pi p/N)\,\varepsilon, \text{ when } k = p, N-p \\ \log(\varepsilon) + \log\left[\pi\sqrt{1 + \left(\dfrac{\sin(2\pi k/N)}{\cos(2\pi p/N) - \cos(2\pi k/N)}\right)^2}\right].\text{ when } k \neq p, N-p \end{cases}$$

As $\log(\varepsilon)$ is the dominant term, to $O(\log(\varepsilon))$, the cepstrum involves finding the ifft of the sequence

$$\tilde{Q}_k = \begin{cases} 0, \text{ when } k = p, N-p \\ \log(\varepsilon) \text{ when } k \neq p, N-p \end{cases}$$

One can subtract any constant from a sequence without affecting any term of its ifft apart from the DC term. We chosse to subtract $\log(\varepsilon)$ from $\tilde{Q}_k$ which gives us the sequence,

$$\tilde{F}_k = \begin{cases} -\log(\varepsilon), \text{ when } k = p, N-p \\ 0.\text{ when } k \neq p, N-p \end{cases}$$

The real part of the ifft of $\tilde{F}_k$ is

$$\tilde{C}_k = -2\log(\varepsilon)\cos\left(\frac{2\pi p k}{N}\right)\frac{1}{N}$$

The maximum occurs when $\cos\left(\frac{2\pi p k}{N}\right) = 1$, i.e. when $2\pi k p/N$ is as close as possible to a multiple of 2π, i.e. when k is as close as possible to a multiple of N/p, which is the formula for the quefrency. This formula does not apply when $k = 0$.

7. The cepstrum for sum of two near integer sinusoids

We will assume two sinusoids, one at frequency $p + \varepsilon$, the other at frequency $q + \varepsilon$, where p and q are distinct positive integers less than $N/2$ and ε is small.

Applying Proposition 11, we find that the N-point DFT of $\sin(2(p+\varepsilon)t) + \sin(2\pi(q+\varepsilon)t)$ is

$$F(x)_k = \begin{cases} N j/2 + O(\varepsilon), \text{ when } k = p, N-p \\ N j/2 + O(\varepsilon), \text{ when } k = q, N-q \\ -2\pi\varepsilon + j\dfrac{\pi\varepsilon\sin(2\pi k/N)}{\cos(2\pi p/N) - \cos(2\pi k/N)} + j\dfrac{\pi\varepsilon\sin(2\pi k/N)}{\cos(2\pi q/N) - \cos(2\pi k/N)}, \text{otherwise} \end{cases}$$

The same argument applies as in section (6), that is that the dominant term in $\log(\mathrm{abs}(F(x)_k))$ is $\log(\varepsilon)$. Thus, up to a DC term, to $O(\log(\varepsilon))$, the cepstrum of $\sin(2\,(p+\varepsilon)\,t)+\sin(2\,\pi\,(q+\varepsilon)\,t)$ is the ifft of

$$\tilde{F}_k = \begin{cases} -\log(\varepsilon), \text{ when } k = p, N-p \\ -\log(\varepsilon), \text{when } k = q, N-q \\ 0. \text{otherwise} \end{cases}$$

Denote the k'th term of the real part of the ifft of $\tilde{F}_k$ as $\tilde{C}_k$. Then, for $k = 1, 2, \cdots, N-1$,

$$\tilde{C}_k = -\log(\varepsilon)\left(\cos\left(\frac{2\,\pi\,p\,k}{N}\right) + \cos\left(\frac{2\,\pi\,q\,k}{N}\right)\right)\frac{2}{N}$$

Let r denote an integer that divides both p and q. Let $f = N/r$, the quefrency corresponding to r. Let $p_0 = p/r$ and $q_0 = q/r$. Then

$$\cos\left(\frac{2\,\pi\,p\,f}{N}\right) + \cos\left(\frac{2\,\pi\,q\,f}{N}\right) = \cos(2\,\pi\,p_0) + \cos(2\,\pi\,q_0) = 2$$

which is the maximum value this sum can have. We have therefore proved that

Proposition 12. *The cepstrum of* $\sin(2\,(p+\varepsilon)\,t)+\sin(2\,\pi\,(q+\varepsilon)\,t)$ *attains a maximum at the quefrency* N/r *where* r *is an integer which divides both* p *and* q*.*

References

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3. Mathcad 13 documentation

4. Maxima home page - http://maxima.sourceforge.net